

Interaction between a Stokes wave packet and a solitary wave

D. CLAMOND ^{a*}, J.P. GERMAIN ^b

ABSTRACT. – This paper is a theoretical and experimental study of the propagation of a short gravity wave packet (modulated Stokes wave) over a solitary wave. The theoretical approach used here relies on a nonlinear WKB-type perturbation method. This method yields a theory of gravity waves that can describe both short and long waves simultaneously. We obtain explicit analytical solutions describing the interaction between the soliton and the short wave packet: phase shifts, variations of wavelengths and of frequencies (Doppler effects). In the experimental part of this work the phase shift experienced by the Stokes wave is measured. The theoretical conclusions are confirmed. © Elsevier, Paris.

1. Introduction

The study of the interaction between waves of short and long wavelengths has many applications in various branches of physics where nonlinear dispersive effects are present. The investigation of the interaction between long and short surface waves has already attracted considerable attention. For infinite depth, the works of Henyey *et al.* (1988), Zhang and Melville (1990), Naciri and Mei (1992), Naciri and Mei (1993) and Woodruff and Messiter (1994) are noteworthy. Using various approaches, these authors investigate the interaction between two waves of very different length. A wave is considered to be long when its length is much greater than the depth. Hence, by definition, for infinite depths only short waves of different scales are present. The above investigations, although interesting, cannot therefore be considered to be representative of interaction phenomena between short and long waves. For finite depths, Longuet-Higgins and Stewart (1960), Garrett and Smith (1976), Phillips (1988) and Longuet-Higgins (1987) investigate the interaction between two Stokes waves, one of which has a larger wavelength than the other. As will be shown below, Stokes waves pertain to the short wave theory. As its name indicates, the short wave theory assumes that the waves being studied have small relative wavelengths. Although the resulting solutions can have a long wavelength, the results are farther from the exact solutions than if the wave is short. Thus the assumption that the long wave is of Stokes type makes the conclusions that can be drawn less rigorous than if a ‘real’ long wave were considered, i.e. one resulting from shallow water theory, such as a solitary wave. A mathematical approach of this problem is due to Davey and Stewartson (1974). These authors proposed a system of two coupled equations to investigate the interaction between a Stokes wave and a shallow water long wave. In this system, if the long wave contribution vanishes the system gets reduced to a cubic-Schrödinger equation; this equation describes the evolution of slow modulations of a Stokes wave envelope. On the other hand, if the Stokes wave contribution vanishes, the Davey–Stewartson system reduces to the classical wave equation. The wave equation is able to propagate a long wave but it is not enough to determine its shape: the Davey–Stewartson system is therefore not complete. A complete system of

* Correspondence and reprints.

^a Laboratoire de Sondages Electromagnétiques de l’Environnement Terrestre, Université de Toulon et du Var, BP 132, 83957 La Garde cedex, France.

E-Mail: clamond@isitv.univ-tln.fr.

^b Laboratoire des Ecoulements Géophysiques et Industriels, BP 53, 38041 Grenoble cedex, France.

this type must, at least, include the cubic-Schrödinger equation and a Korteweg–de Vries type equation. Another difficulty of this system is that there is no modification of the long wave due to the short wave. Shen *et al.* (1994) studied numerically the interaction between a Stokes wave of constant amplitude and a solitary wave. These authors found a phase shift of the short wave and a modification of its amplitude when riding on the solitary wave. We will propose below a new interpretation of the latter phenomenon. They however did not find any modification of the soliton, while this feedback must exist.

The aim of this investigation is to study the propagation of a modulated Stokes wave over a solitary wave, i.e. one wave belonging to the short wave theory and the other to the shallow water theory. In this way the field of investigations is widened. In section 2, we present the equations and the statement of the problem. Section 3 is devoted to a simplified analysis of the interaction between a Stokes wave and a solitary wave. This simplified analysis predicts a phase shift of the Stokes wave. Next, in section 4, we give the perturbation technique used for the theoretical study which is developed in section 5. The approximation of the solution, presented in section 6, predicts a phase shift of the Stokes wave and phase modifications of the long wave with vertical variations (section 7). In section 8 we study the Doppler effects on the short and long waves. The theoretical and simplified formulae of the phase shift of the Stokes wave are compared with experiment in section 9.

2. Statement of the problem

2.1. EQUATIONS

We wish to describe surface gravity waves in a homogeneous, non-viscous, incompressible fluid subject to the acceleration due to gravity g , and we neglect surface tension. The flow is assumed to be two dimensional and irrotational. The investigation is carried out in the reference frame of zero mean flow. x , y and t denote the horizontal, upward vertical and time variables respectively; the origin of the axis is chosen to be in the plane of the average level. The bottom is horizontal, impermeable and located at position $y = -h$. The difference in height between the free surface and the mean level $y = 0$ is denoted by η . The surface is assumed to be impermeable and the pressure is constant. The irrotational property of the flow implies the existence of a velocity potential φ . The incompressibility of the fluid expressed in terms of the velocity potential, the impermeability of the bottom, the isobaric condition and impermeability of the free surface, are written

$$\varphi_{xx} + \varphi_{yy} = 0 \quad \text{for} \quad -h \leq y \leq \eta, \quad (1)$$

$$\varphi_y = 0 \quad \text{at} \quad y = -h, \quad (2)$$

$$2g\eta + 2\varphi_t + \varphi_x^2 + \varphi_y^2 = 0 \quad \text{at} \quad y = \eta, \quad (3)$$

$$\varphi_y - \eta_t - \eta_x \varphi_x = 0 \quad \text{at} \quad y = \eta. \quad (4)$$

These equations are to be solved in a region where one of the boundaries, the free surface, is unknown. To overcome this first difficulty, the relations at the free surface are expressed in terms of the rest position through a Taylor expansion around $y = 0$. A second obstacle is the non-linearity of the equations for the free surface. Generally, to obtain approximate analytical expressions, perturbation techniques are employed. There are two main theories based on these techniques.

2.2. CLASSICAL THEORIES

One technique, Poincaré's small parameter method, consists of looking for a solution near the rest position that can be expanded as an integral power series in a parameter ε (Stoker, 1957). The first order approximation

gives rise to the linearised theory, the solutions of which can be expressed in terms of circular and hyperbolic functions. At higher orders, inhomogeneous linear equations must be solved, and it is found that each order adds another harmonic. This yields a solution in the form of a Fourier series. The results are such that when the wavelength is allowed to tend to infinity, in order for the solutions to remain finite, the amplitude must be zero. For this reason these solutions are described as short linear waves, and the theory is also known as that of short waves. It turns out that Stokes theory is, in particular, incapable of describing solitary waves.

Confronted with the inability of Poincaré's small parameter method to deal with long waves of significant amplitude, and in particular solitary waves, a second theory can be invoked, that of shallow water. To allow for large scales, a distortion is introduced, characterized by a parameter ε , in the horizontal and time variables (Germain, 1967). A solution lying close to the rest position is then sought that can be expanded as an integral power series in ε . For progressive waves this yields cnoidal solutions (so called because they are written in terms of Jacobi's cn-function) which admit solitary waves in the limiting case. In the limiting case where the elliptic functions reduce to circular functions, the amplitude of the wave vanishes. This theory therefore cannot describe the solutions of the previous one. It is also known as the theory of long waves.

2.3. LITTMAN'S DIAGRAM

We have just seen that these two theories have different ranges of validity, as can be seen from the qualitative diagram of Littman (1957, figure 1). A more detailed investigation can be found in Komar (1976). This diagram is qualitative in that the exact boundaries are unknown for the range of validity of each theory. Mathematically, Poincaré's small parameter method generates a regular perturbation problem: the resulting series are convergent (Levi-Civita, 1925). In contrast, the technique used in the shallow water theory leads to a singular perturbation problem: the resulting series are divergent (Germain, 1967). It is not possible to pass from the solutions of one theory (at a given order of approximation) to those of the other.

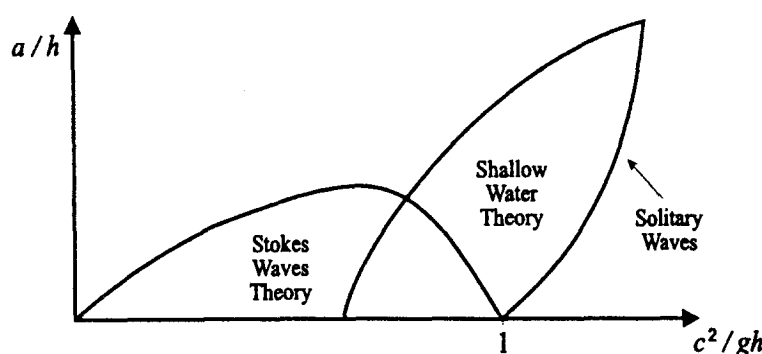


Fig. 1. – Littman's diagram.

The purpose of this investigation is to study the interaction between a Stokes wave of constant amplitude and a solitary wave. As we have just seen, these waves stem from two distinct theories that do not share the same range of validity. It clearly follows that we can use neither of these theories to solve our problem. We have to turn to a more general calculation technique.

3. A simplified approach

Before starting the theoretical investigation we can use physical reasoning to get a first idea of what happens when a solitary wave interacts with a constant amplitude Stokes wave of wavelength Λ ($\Lambda = 2\pi/k$, where

k is the wavenumber). For this, consider the trajectories of fluid particles subjected to a soliton. The effect of a solitary wave is to displace a fluid particle horizontally by a distance L : the solitary wave produces a net displacement of matter. By expressing first-order shallow water theory in terms of Lagrangian coordinates, one gets for the displacement L

$$L = \frac{4h}{\sqrt{3}} \sqrt{\frac{\hat{a}}{h}},$$

where \hat{a} is the amplitude of the soliton. If there is an interaction between the soliton and the short wave, the displacements due to the two waves combine. If we assume that this combination is a simple algebraic sum of the displacements, the short wave is translated by L after the passage of the solitary wave. The Stokes wave has therefore undergone a phase shift equal to

$$\Delta\phi = 2\pi \frac{L}{\Lambda} = \frac{4kh}{\sqrt{3}} \sqrt{\frac{\hat{a}}{h}}. \tag{5}$$

(5) is an approximate expression for the phase shift experienced by the short wave as a function of the amplitude of the soliton. It is obtained by considering an algebraic superposition of the displacements. We define the limits of validity of this hypothesis in section 7. (5) states that the phase shift is proportional to the square root of the soliton amplitude and to the wave number of the short wave. The amplitude of the short wave does not itself appear in this formula. It can also be seen that the phase shift is the same whether the two waves move in the same direction or in opposite directions.

4. Method of Solution

In order to overcome the incompatibility of existing theories, we use a perturbation technique that yields a description of the two types of wave simultaneously. Nevertheless, instead of treating the problem in complete generality, we limit ourselves to the interaction between a Stokes wave with sinusoidal modulations of its amplitude and a solitary wave. Much more clearly than a complete treatment, this limited investigation reveals the spirit of the method, the calculation techniques and the role played by the various terms.

As in the shallow water theory, we introduce a distortion in the horizontal and time variables using an unspecified parameter ε

$$\alpha = \varepsilon x, \quad \tau = \varepsilon t.$$

α and τ are sometimes called ‘slow’ variables, as opposed to x and t , which are designated as ‘fast’ variables. We look for solutions, close to the rest position, that can be expanded as an integral power series in the parameter ε . Thus we set

$$\varphi = \sum_{n=1}^{\infty} \varepsilon^n \varphi_n(\alpha, y, \tau), \quad \eta = \sum_{n=1}^{\infty} \varepsilon^n \eta_n(\alpha, \tau).$$

At present, this solution method resembles shallow water theory. We therefore have to generalize these series so that they can describe a short wave. The theory of short waves shows that the latter can be approximated by Fourier series. We therefore introduce such series in our search for solutions (i.e. in the expression for φ_n and η_n). The short wave is introduced at order ε^2 . This assumption has a distinct advantage: it greatly simplifies the calculations. It can be shown that, for a Stokes wave of constant amplitude and for a Stokes wave of

sinusoidal modulations, this consideration in no way restricts the generality of the solution, at least for the order of approximation that we are considering here (Clamond, 1994).

In paragraph 2.2 it was seen that the solution technique of the Stokes theory generates a fundamental frequency, and that each order introduces a new harmonic. Since we start the expansions here at order ε^2 , it follows that only even powers of ε will introduce a new harmonic. According to the same theory, short waves possess a phase of the form $(kx + \omega t) = (k\alpha + \omega\tau)/\varepsilon$ (where k is the wavenumber and ω the angular frequency). We therefore use ‘Fourier series’ expansions in which the phase has the form $S(\alpha, \tau)/\varepsilon$. The handling of the calculations is simplified by expressing these series in their complex form. These considerations naturally lead us to expand φ_n and η_n as trigonometric polynomials of the form

$$\varphi_n = \sum_{p=-[n/2]}^{[n/2]} F_{n,p}(\alpha, \tau) \exp\left(\frac{ipS}{\varepsilon}\right), \quad \eta_n = \sum_{p=-[n/2]}^{[n/2]} G_{n,p}(\alpha, \tau) \exp\left(\frac{ipS}{\varepsilon}\right), \quad (6)$$

where $[]$ denotes the integral part of $n/2$, and $i^2 = -1$. In relations (6), if only the terms $p = 0$ are retained, then the formulation of shallow water theory is recovered. Similarly, if we require $F_{n,p}$ and $G_{n,p}$ to be independent of both α and τ , and if $S = k\alpha + \omega\tau$, then we recover the Stokes wave. This theory contains the previously existing ones. Relations (6) mean that we look for short waves possessing a ‘slow’ variation in amplitude and a ‘fast’ variation in phase. To take account also of the ‘slow’ phase variations, S must be expanded also. Thus

$$S = \sum_{n=0}^{\infty} \varepsilon^n S_n(\alpha, \tau). \quad (7)$$

It can be seen that we are in fact only using so-called ‘ray’ theory. This has merely been generalized by introducing ‘harmonics’ to account for the non-linearity of the equations. The straightforward case $S = S_0$ is generally called the ‘geometrical optics approximation’, and the case $S = S_0 + \varepsilon S_1$ the ‘physical optics approximation’. We have founded this solution method on physical considerations, seeking inspiration from existing theories. The argument can also be developed from a purely mathematical point of view. The distortion of the variables makes the equations singular. In order to calculate the regular solutions of a distorted equation, a technique exists that has been known for a long time: the classic WKB method (Bender and Orszag, 1978). The introduction of ‘harmonics’ slightly modifies this technique, so that account can be taken of the non-linearity of our equations. The WKB-type technique used here is not new. It was introduced already, in a similar form, by Chu and Mei (1970). Those authors use it to investigate modifications to the envelope and phase of a three dimensional Stokes wave over a variable bottom. This work is very general, and it seems that its authors did not recognize the possibility, inherent in the method, of describing both short and long waves simultaneously. The solution method has the drawback of algebraic complexity in establishing the equations (see Appendix). This may explain why Chu and Mei did not realized its potential for describing long waves. We have therefore chosen to show the detailed calculations so as to reveal the physical role played by the different terms. Nonlinear WKB-type methods were used by several authors for various investigations. In all papers we have read, this technique was never used to investigate the interaction between two phenomena described by regular and singular perturbation method.

The equations must hold for arbitrary ε , they must do so for each of its powers independently. The factors $\exp(ipS/\varepsilon)$ are assumed to be linearly mutually independent: the equations are therefore valid for each ‘harmonic’ independently. Although we use complex notation, φ and η are real, thus

$$F_{n,-p} = F_{n,p}^*, \quad G_{n,-p} = G_{n,p}^*,$$

where \star denotes the complex conjugate. It therefore suffices to solve the equations for the harmonics of positive rank, those of negative rank being deduced by conjugation. Since $F_{n,p}$ and $G_{n,p}$ are of order ε^0 , the same is true for their arguments. Thus, with a suitable change of notation, S_1 can be included in the arguments of $F_{n,p}$ and $G_{n,p}$. This shows that S_1 can, quite generally, be taken to be zero (i.e. $S_1 \equiv 0$). To simplify the notation, we write

$$k \equiv \frac{\partial S_0}{\partial \alpha}, \quad \omega \equiv \frac{\partial S_0}{\partial \tau}, \quad \Longleftrightarrow \quad S_0 \equiv k\alpha + \omega\tau + \tilde{\delta},$$

where k and ω are the (constant) wavenumber and angular frequency of the short wave, and $\tilde{\delta}$ is a constant phase shift. With no loss of generality to the solution, we take k to be positive; ω , however, is algebraic. These parameters describe the overall characteristics of the Stokes wave. Moreover, let $\tilde{c} = \omega/k$ be the short wave phase speed.

Lastly, we require physically acceptable solutions, i.e. regular and bounded in the range $(-\infty \leq \alpha \leq +\infty; -h \leq y \leq \eta; -\infty \leq \tau \leq +\infty)$. Thus if secular terms arise in the solutions, they must be set equal to zero. Note that the bounded character of the solutions applies only to their moduli; neither their arguments nor S are necessarily so constrained.

5. Solution of the equations

The condition of impermeability at the bottom yields, for all powers of ε and for all harmonics,

$$\frac{\partial F_{n,p}}{\partial y} = 0, \quad \text{at} \quad y = -h. \quad (8)$$

The Laplace equation at orders ε and ε^2 yields

$$\begin{aligned} \frac{\partial^2 F_{1,0}}{\partial y^2} = 0 &\implies F_{1,0} = A_{1,0}(\alpha, \tau), \\ \frac{\partial^2 F_{2,0}}{\partial y^2} = 0 &\implies F_{2,0} = A_{2,0}(\alpha, \tau), \\ \frac{\partial^2 F_{2,1}}{\partial y^2} - k^2 F_{2,1} = 0 &\implies F_{2,1} = A_{2,1}(\alpha, \tau) \cosh k(y+h). \end{aligned}$$

The Bernoulli equation of order ε^2 gives

$$G_{2,0} = -\frac{1}{g} \frac{\partial A_{1,0}}{\partial \tau}, \quad G_{2,1} = -\frac{i\omega}{g} \cosh kh A_{2,1},$$

and the surface impermeability yields Airy's dispersion relation

$$\omega^2 = gk \tanh kh. \quad (9)$$

As far as the 'fast' variation of the short wave and the dispersion relation are concerned, we recover the results of linear theory. To obtain the 'slow' variations of phase and amplitude, as well as the soliton, the higher orders must be solved.

Unfortunately, the equations of higher orders do not give such simple expressions. Hence, we focus here our attention on the resolution of essential equations. The solutions of the Laplace equation (which gives the $F_{n,p}$),

and of the Bernoulli equation (which gives the $G_{n,p}$), and the equations directly derivable from the surface impermeability are given in the appendix.

5.1. SOLUTION OF ORDER ε^3

The surface impermeability yields the wave equations

$$\hat{c}^2 \frac{\partial^2 A_{1,0}}{\partial \alpha^2} - \frac{\partial^2 A_{1,0}}{\partial \tau^2} = 0, \quad (10)$$

$$\tilde{c}_g \frac{\partial A_{2,1}}{\partial \alpha} - \frac{\partial A_{2,1}}{\partial \tau} = 0, \quad (11)$$

with

$$\hat{c} = \sqrt{gh}, \quad \tilde{c}_g = \frac{d\omega}{dk} = \frac{\tilde{c}}{2} \left(1 + \frac{2kh}{\sinh 2kh} \right), \quad (12)$$

\hat{c} being the phase velocity of the long wave and \tilde{c}_g being the group velocity of the short wave. It can be seen in (10) that $A_{1,0}$ describes a long wave moving in the direction of decreasing α at velocity \hat{c} , and one moving towards increasing α at the same velocity. Since we wish to describe only one long wave coming from $\alpha = -\infty$, we restrict the solution of (10) to the particular case

$$A_{1,0} = A_{1,0}(\alpha - \hat{c}\tau). \quad (13)$$

The most general solutions of (11) is

$$A_{2,1} = A_{2,1}(\alpha + \tilde{c}_g\tau). \quad (14)$$

Relation (14) states that the envelope of the short wave is a wave propagating at the group velocity (i.e. a standard result). At this order the classical theories of Stokes and of shallow water are recovered, but our knowledge of the solution we seek is not yet complete. We must therefore solve the next order.

5.2. SOLUTION OF ORDER ε^4

The impermeability of the free surface for the harmonics of rank 0 and 2 gives in turn

$$\hat{c}^2 \frac{\partial^2 A_{2,0}}{\partial \alpha^2} - \frac{\partial^2 A_{2,0}}{\partial \tau^2} = 0, \quad (15)$$

$$A_{4,2} = -\frac{3ik^2}{4\omega \sinh^2 kh} A_{2,1}^2. \quad (16)$$

Relation (16) states that the first harmonic is completely determined by the fundamental; (16) is identical to the relation obtained from the second order Stokes theory. Like $F_{1,0}$, $F_{2,0}$ must describe the same long wave coming from $\alpha = -\infty$. Relation (15) states that $F_{2,0}$ is not forced by $F_{1,0}$, so $F_{2,0}$ can be taken equal to zero without loss of generality. Lastly, from the impermeability of the free surface for the harmonic of rank 1 we obtain the equation (A.1) (see Appendix). Relations (13) and (14) suggest a change of independent variables

$$\mu = \alpha + \tilde{c}_g\tau, \quad \nu = \alpha - \hat{c}\tau.$$

In terms of the variables μ and ν , and after an integration with respect to ν , equation (A.1) yields

$$iA_{3,1} = S_2 A_{2,1} - \frac{k(\hat{c} - \tilde{c}_g + \tilde{c}/2)}{\hat{c}^2 + \hat{c}\tilde{c}_g} A_{1,0} A_{2,1} - \frac{\nu}{2} \frac{\omega_{kk}}{\hat{c} + \tilde{c}_g} \frac{d^2 A_{2,1}}{d\mu^2} + i\Xi_1(\mu), \quad (17)$$

where Ξ_1 is an arbitrary complex function of μ and

$$\omega_{kk} = \frac{d^2 \omega}{dk^2} = \frac{\hat{c}^2}{\omega} - \frac{\tilde{c}_g^2}{\omega} - 2h\tilde{c}_g \tanh kh.$$

We now set

$$A_{2,1} = a_2 \exp i\psi, \quad A_{3,1} = a_3 \exp i\psi, \quad \Xi_1 = \xi_1 \exp i\psi,$$

where a_2 , a_3 , ξ_1 and ψ are real functions of μ . Multiplying (17) by $\exp -i\psi$, then taking the imaginary part, we get

$$a_3 = -\frac{\nu}{2} \frac{\omega_{kk}}{\hat{c} + \tilde{c}_g} \left(a_2 \frac{d^2 \psi}{d\mu^2} + 2 \frac{da_2}{d\mu} \frac{d\psi}{d\mu} \right) + \xi_1. \quad (18)$$

It can be seen from (18) that a_3 is a linear function of the variable ν : this is a secular term. For the solution to be physically acceptable, a_3 must be bounded. This consideration leads us to the condition of non-secularity

$$a_2 \frac{d^2 \psi}{d\mu^2} + 2 \frac{da_2}{d\mu} \frac{d\psi}{d\mu} = 0 \quad \Rightarrow \quad \frac{d\psi}{d\mu} = h\hat{c}^2 \gamma_1 a_2^{-2}, \quad (19)$$

where γ_1 is an arbitrary real constant. Hence (18) yields that a_3 and $A_{3,1}$ are only functions of μ . Multiplying (17) by $\exp -i\psi$, then taking the real part and using (19), we find

$$S_2 = \frac{k(\hat{c} - \tilde{c}_g + \tilde{c}/2)}{\hat{c}^2 + \hat{c}\tilde{c}_g} A_{1,0} + \frac{\nu}{2} \frac{\omega_{kk}}{\hat{c} + \tilde{c}_g} \left(\frac{1}{a_2} \frac{d^2 a_2}{d\mu^2} - \frac{h^2 \hat{c}^4 \gamma_1^2}{a_2^4} \right) = \tilde{S}_2(\nu) + \frac{\nu}{h} R_2(\mu), \quad (20)$$

where \tilde{S}_2 and R_2 are introduced to simplify the expressions in subsequent equations. Equation (20) fixes the way in which the long wave modifies the phase of the short wave. At present we have specified the role played by the harmonic of rank zero (description of the solitary wave). We have also calculated the harmonic of rank two (first harmonic) as a function of that of rank one (fundamental). Lastly, the 'slow' variations of phase and amplitude have been related to each other, as well as the phase changes of the short wave produced by the solitary wave. Nevertheless, since we still do not have complete knowledge of the first-order approximation either of the short wave or of the soliton, we must continue to solve for the next higher order.

5.3. SOLUTION OF ORDER ε^5

The impermeability of the free surface for the harmonic of rank 0 (A.2), with the variables μ and ν and after integration with respect to μ gives

$$\begin{aligned} (\tilde{c}_g^2 - \hat{c}^2) \frac{\partial A_{3,0}}{\partial \mu} - 2\hat{c}(\tilde{c}_g + \hat{c}) \frac{\partial A_{3,0}}{\partial \nu} &= \mu \left[\frac{gh^3}{3} \frac{d^4 A_{1,0}}{d\nu^4} + 3\hat{c} \frac{dA_{1,0}}{d\nu} \frac{d^2 A_{1,0}}{d\nu^2} \right] \\ &\quad - k^2 (\tilde{c}_g + 2\tilde{c} \cosh^2 kh) |A_{2,1}|^2. \end{aligned} \quad (21)$$

It can be seen that (21) contains a secular term in μ . For the solution to be physically acceptable, this secular term must be zero. We therefore set

$$\frac{d^4 A_{1,0}}{d\nu^4} + \frac{9}{\hat{c}h^2} \frac{dA_{1,0}}{d\nu} \frac{d^2 A_{1,0}}{d\nu^2} = 0. \quad (22)$$

In (22) we have recovered the classical equation for long waves as found in the shallow water theory. The most general solution can be expressed with Jacobi's elliptic functions; these solutions are known as cnoidal waves. This solution allows as a particular solution the solitary wave of the form

$$A_{1,0} = -\frac{\hat{c}\gamma_2^2}{3}(\nu - \nu_0) + \frac{2h\hat{c}\gamma_2}{\sqrt{3}} \tanh \left[\frac{\sqrt{3}}{2h} \gamma_2(\nu - \nu_0) \right], \quad (23)$$

where γ_2 and ν_0 are real arbitrary constants. It can be seen that in first approximation the solitary wave is unaffected by the short wave, i.e. the short wave does not enter into the determination of $A_{1,0}$. We also observe that $A_{1,0}$ is unbounded; we shall see in section 6 how to avoid it by renormalization.

The relation (21) then gives for $A_{3,0}$

$$A_{3,0} = \frac{k^2}{h\omega} \frac{h\omega\hat{c}_g + \hat{c}^2 \sinh 2kh}{\hat{c}^2 - \hat{c}_g^2} \int |A_{2,1}|^2 d\mu. \quad (24)$$

This relation was first obtained by Benney and Roskes (1969).

The condition of impermeability at the free surface for the harmonic of rank 1 (A.3), with the variables μ and ν and after integration with respect to ν gives

$$\begin{aligned} A_{4,1} = & \Xi_2 - \frac{\hat{\Gamma}}{\hat{c}} A_{2,1} \frac{dA_{1,0}}{d\nu} + \frac{\bar{\Gamma}}{\hat{c}} \frac{dA_{2,1}}{d\mu} A_{1,0} - iS_3 A_{2,1} + \nu P \\ & - \frac{\nu^2}{4h} \frac{\omega_{kk}}{\hat{c} + \hat{c}_g} \left(A_{2,1} \frac{d^2 R_2}{d\mu^2} + 2 \frac{dA_{2,1}}{d\mu} \frac{dR_2}{d\mu} \right), \end{aligned} \quad (25)$$

where Ξ_2 is an arbitrary complex function of μ , P a function of μ (see Appendix) and

$$\hat{\Gamma} = \frac{\hat{c}(\hat{c} - \hat{c}/2)}{\hat{c}(\hat{c} + \hat{c}_g)} + \frac{h^2\omega^2(4\hat{c} + \hat{c})}{2\hat{c}(\hat{c} + \hat{c}_g)^2}, \quad \bar{\Gamma} = \frac{\hat{c}/2}{\hat{c} + \hat{c}_g} + \frac{h^2\omega^2\hat{c}\hat{c}_g/\hat{c}^2 - 2k\hat{c}\omega_{kk}}{(\hat{c} + \hat{c}_g)^2}.$$

Now set

$$A_{4,1} = a_4 \exp i\psi, \quad \Xi_2 = \xi_2 \exp i\psi, \quad P = p \exp i\psi,$$

then multiply relation (25) by $\exp -i\psi$ and take the real part. Thus

$$a_4 = \xi_2 - \frac{\hat{\Gamma}}{\hat{c}} a_2 \frac{dA_{1,0}}{d\nu} + \frac{\bar{\Gamma}}{\hat{c}} A_{1,0} \frac{da_2}{d\mu} + \nu p - \frac{\nu^2}{4h} \frac{\omega_{kk}}{\hat{c} + \hat{c}_g} \left(a_2 \frac{d^2 R_2}{d\mu^2} + 2 \frac{da_2}{d\mu} \frac{dR_2}{d\mu} \right). \quad (26)$$

The condition of non-secularity in ν^2 for (26) gives

$$a_2 \frac{d^2 R_2}{d\mu^2} + 2 \frac{da_2}{d\mu} \frac{dR_2}{d\mu} = 0, \quad \Rightarrow \quad \frac{dR_2}{d\mu} = h\hat{c}^2 \gamma_3 a_2^{-2}, \quad (27)$$

where γ_3 is a real constant. Using the definition of R_2 and integrating with respect to μ we get

$$a_2 \frac{d^2 a_2}{d\mu^2} - \left(\frac{da_2}{d\mu} \right)^2 - 2 \frac{h^2 \hat{c}^4 \gamma_1^2}{a_2^2} - 2 \frac{\hat{c} + \hat{c}_g}{\omega_{kk}} \hat{c}^2 \gamma_3 \mu = \text{constant}.$$

A new non-secularity condition then requires that

$$\gamma_3 = 0 \quad \Rightarrow \quad R_2 = \text{constant} = 0.$$

Introduction of shifts of order ε^2 in the definitions of k , ω and of the origin allows R_2 to be set equal to zero in the general case. Relation (27) is then integrated in the form

$$a_2^2 \left(\frac{da_2}{d\mu} \right)^2 + h^{-2} \gamma_4 a_2^4 - \hat{c}^2 \gamma_5 a_2^2 + h^2 \hat{c}^4 \gamma_1^2 = 0, \quad (28)$$

where the γ_n are real arbitrary constants. (28) is a differential equation determining the ‘slow’ variation of the amplitude of the short wave. This equation is identical to the permanent form of the linear Schrödinger equation. It can be seen from (28) that, to first order, the long wave is not involved in the determination of the amplitude of the short wave. (28) has real bounded solutions only if

$$\gamma_4 \geq 0 \quad \text{and} \quad \gamma_5 \geq 2|\gamma_1|\sqrt{\gamma_4},$$

and the general solution is

$$a_2 = h\hat{c}\sqrt{\gamma_5/2\gamma_4} \sqrt{1 + \sqrt{1 - 4\gamma_1^2\gamma_4/\gamma_5^2} \sin[2\sqrt{\gamma_4}(\mu - \mu_0)/h]}. \quad (29)$$

Relation (19) then gives the expression of ψ

$$\psi = \arctan \left\{ \frac{\sqrt{1 - 4\gamma_1^2\gamma_4/\gamma_5^2} + \tan[2\sqrt{\gamma_4}(\mu - \mu_0)/h]}{2\gamma_1\sqrt{\gamma_4}/\gamma_5} \right\}. \quad (30)$$

We have now obtained the ‘slow’ modifications of the Stokes wave. To obtain a Stokes wave of constant amplitude, we set $\gamma_1 = \gamma_4 = \gamma_5 = 0$.

(26) now becomes

$$a_4 = \xi_2 - \frac{\hat{\Gamma}}{\hat{c}} a_2 \frac{dA_{1,0}}{d\nu} + \frac{\bar{\Gamma}}{\hat{c}} A_{1,0} \frac{da_2}{d\mu}. \quad (31)$$

This relation seems to describe how the amplitude of the short wave is modified by the long wave. This result is in agreement with the study of Shen *et al.* (1994). We shall see that, in fact, it is possible to perform this as the modification of the solitary wave and as the modification of the short wave envelope.

6. Renormalizations of the theoretical results

As a first approximation, we may take the solution limited to the order ε^2

$$\varphi \simeq \varepsilon \varphi_1 + \varepsilon^2 \varphi_2, \quad \eta \simeq \varepsilon^2 \eta_2, \quad S \simeq S_0 + \varepsilon^2 S_2.$$

As it stands, this solution is not applicable: the velocity potential is unbounded, and the mean level does not coincide with the rest level. These solutions must now be expressed in terms of measurable parameters.

6.1. MEAN LEVEL

We wish to express our solutions as a function of the mean height h^* , since this is the quantity that is measured experimentally. It is given by

$$h^* = \langle h + \eta \rangle = \lim_{M, N \rightarrow \infty} \frac{1}{4MN} \int_{-M}^{+M} \int_{-N}^{+N} (h + \eta) \, dx \, dt = h - \frac{\varepsilon^2 \gamma_2^2 h}{3} = h - \Delta h.$$

The mean level clearly does not coincide with the axis $y = 0$. More specifically, the mean level coincides with the axis $y = 0$ at order ε^0 , but not at order ε^2 . Physically, the mean level is also the rest level: therefore h^* must enter the solutions. In order to compare our solutions with experiments we must introduce a change of axes of the form

$$y^* = y + \Delta h \quad \implies \quad \eta^* = \eta + \Delta h.$$

Having fixed the origin of our system of axes, we introduce three measurable quantities: the amplitudes of the soliton and of the short wave, \hat{a} and \tilde{a} . These are defined as the maxima of both waves with respect to the mean level. We also introduce the eccentricity of the envelope e . Thus

$$\hat{a} = 2\varepsilon^2 h \omega \sqrt{h/g} \cosh kh \sqrt{2\gamma_5/\gamma_4}, \quad \hat{a} = \varepsilon^2 h \gamma_2^2 \quad e = \sqrt{1 - 4\gamma_1^2 \gamma_4/\gamma_5^2}.$$

The approximations of parameters were determined to order ε^2 , hence the introduction of h^* allows us to keep the approximations of order ε^2 , since the higher orders are not significant. For example

$$\sqrt{gh} = \sqrt{gh^*} [1 + \varepsilon^2 \gamma_2^2/6 + O(\varepsilon^4)].$$

We must now determine in what frame the solutions were obtained.

6.2. MEAN FLOW

The mean flow, $\langle q \rangle$, is defined by

$$\langle q \rangle = \left\langle \int_{-h}^{\eta} \varphi_x \, dy \right\rangle = -\frac{\varepsilon^2 \gamma_2^2}{3} h \sqrt{gh} + O(\varepsilon^4).$$

For the frame to which our calculations have led us, it turns out that the mean flow is not zero. The flow velocity is

$$\langle u \rangle = \frac{\langle q \rangle}{h^*} = -\frac{\varepsilon^2 \gamma_2^2}{3} \sqrt{gh^*} + O(\varepsilon^4).$$

Experimentally, velocities are determined in the laboratory frame. This is also the frame in which the mean velocity is zero. Hence, to compare our solutions with experiments, we must express them in the frame of nil mean flow. For this we apply the following Galilean transformation

$$x = x^* + \langle u \rangle t, \quad \varphi = \varphi^* + \langle u \rangle x^* + \frac{\langle u \rangle^2}{2} t, \quad \hat{c} = \hat{c}^* + \langle u \rangle, \quad \tilde{c} = \tilde{c}^* + \langle u \rangle, \quad \tilde{c}_g = \tilde{c}_g^* + \langle u \rangle.$$

Changes of order ε^2 in the definitions of k and ω allow the linear term in v (resulting from the Galilean transformation) to cancel out in the expression for \tilde{S} .

6.3. REMARK

The solution technique yielded solutions in a reference frame that is not that of a zero mean flow, and whose mean level does not coincide with the rest level. This contradicts our initial assumptions. Such a phenomenon is not peculiar to this solution technique. It arises with the majority of perturbation methods. When it does not show up, this is often because it enters at a higher order than that of the approximation considered (generally the first). It can be avoided by introducing multiple scales. This technique is possible for the method we have used, but it complicates the algebra in the equations. Furthermore, it is not certain that it completely removes the difficulties with the mean level and flow. The changes in definition of the parameters, the purpose of which was to introduce physical quantities, are called renormalization and are very useful in theoretical physics.

6.4. RENORMALIZATION OF THE LONG WAVE PHASE

The analytical solution shows that the soliton is not affected by the short wave, while the latter undergoes a important modification in phase through the function $\hat{\phi}$. This solution is not complete: if the long wave modifies the short wave, symmetrically the short wave modifies the long wave; there is necessary a feed back. In the perturbation series, we expanded the phase of the short wave. *A priori* the phase of the long wave must also be expanded by the strained co-ordinates technique (Nayfeh, 1973). This expansion considerably increases the resolution difficulty and the calculus becomes practically impossible. In fact, it is not necessary to use this technique directly, it is possible to recover the results by the Pritulo–Crocco renormalization (Pritulo, 1962; Crocco, 1972). If we consider that the long wave is described by the function

$$\varepsilon A_{1,0} \equiv \varepsilon A_{1,0}(\hat{S}), \quad \hat{S} = \varepsilon[x - \hat{c}t + \varepsilon^2 R(x, y, t) + \dots],$$

then a Taylor expansion of this expression gives

$$\varepsilon A_{1,0} = \varepsilon A_{1,0}(\varepsilon x - \varepsilon \hat{c}t) + \varepsilon^4 R(x, y, t) A'_{1,0}(\varepsilon x - \varepsilon \hat{c}t) + O(\varepsilon^5).$$

The term $RA'_{1,0}$ is similar to the term $\hat{\Gamma}a_2A'_{1,0}$ in (31). It is therefore natural to introduce this last term back in the phase of the long wave. This interaction term which first seemed to describe the modification of the short wave amplitude, now appears to be the modification of the long wave by the Stokes wave; the physics is drastically different. Shen *et al.* found a modification of the amplitude of the Stokes wave because they assumed that the solitary wave is not modified, so they implicitly made a Taylor expansion of the soliton's phase. Note that this type of renormalization is also called a Lie group transformation.

6.5. RENORMALIZATION OF THE ENVELOPE WAVE PHASE

We found that the long wave modifies the phase of the carrier wave but not the phase of its envelope. As for the long wave and the carrier wave, the phase of the envelope wave must be modified in the interaction. A similar reasoning as previously, leads us to consider the term $\bar{\Gamma}/\hat{c}A_{1,0}da_2/d\mu$ in (31), as an expansion of the envelope's phase. Hence, we introduce it back in the phase of a_2 , and we obtain in this way the modification of the short wave envelope due to the long wave.

After the renormalizations, the relation (31) becomes

$$a_4 = \xi_2(\mu),$$

hence we conclude that there is no modification of the amplitude of the Stokes wave.

6.6. FINAL SOLUTION

To simplify the expressions, we omit the * on the parameters. Throughout the rest of this article we consider only the quantities redefined after renormalization. Finally the solution can be written

$$\varphi = \frac{g\tilde{a}}{\omega \cosh kh} \cosh k(y+h) \sqrt{1+e \sin 2\bar{S}} \cos(\tilde{S} + \psi) + \frac{2h\sqrt{g\tilde{a}}}{\sqrt{3}} \tanh \hat{S},$$

$$\eta = \tilde{a} \sqrt{1+e \sin 2\bar{S}} \sin(\tilde{S} + \psi) + \hat{a} \operatorname{sech}^2 \hat{S}_0, \quad \psi = \arctan \left[\frac{e + \tan \bar{S}}{\sqrt{1-e^2}} \right],$$

with the phase functions

$$\begin{aligned} \tilde{S} &= \theta + \tilde{\phi}(v), & \bar{S} &= \Theta + \bar{\phi}(v), & \hat{S} &= v - \hat{\phi}(\theta, \Theta, y), & \hat{S}_0 &= \hat{S}(y=0), \\ \theta &= kx + \omega t + \tilde{\delta}, & \Theta &= Kx + \Omega t + \bar{\delta}, & v &= \kappa x - \sigma t + \hat{\delta}, \end{aligned}$$

where $\tilde{\delta}$, $\bar{\delta}$ and $\hat{\delta}$ are constant phase shifts, and phase perturbation functions are

$$\begin{aligned} \tilde{\phi} &= \frac{2kh\tilde{\Gamma}}{\sqrt{3}} \sqrt{\frac{\tilde{a}}{h}} \tanh v, & \bar{\phi} &= \frac{2Kh\bar{\Gamma}}{\sqrt{3}} \sqrt{\frac{\hat{a}}{h}} \tanh v, \\ \hat{\phi} &= \frac{\tilde{a}g\kappa\hat{\Gamma}}{c_0\omega \cosh kh} \sqrt{1+e \sin 2\Theta} \cosh k(y+h) \cos \left[\theta + \arctan \left(\frac{e + \tan \Theta}{\sqrt{1-e^2}} \right) \right], \end{aligned}$$

and the relations between parameters are

$$\begin{aligned} \tilde{\Gamma} &= \frac{c_0 - \tilde{c}_g + \tilde{c}/2}{c_0 + \tilde{c}_g}, & \bar{\Gamma} &= \frac{c_0(\tilde{c} + 4\tilde{c}_g - 4c_0)}{2\tilde{c}(c_0 + \tilde{c}_g)} + \frac{h^2\omega^2\tilde{c}_g(4c_0 + \tilde{c})}{c_0^2(c_0 + \tilde{c}_g)^2}, \\ \hat{\Gamma} &= \frac{c_0(c_0 - \tilde{c}/2)}{\tilde{c}(c_0 + \tilde{c}_g)} + \frac{h^2\omega^2(4c_0 + \tilde{c})}{2c_0(c_0 + \tilde{c}_g)^2}, & \omega^2 &= gk \tanh kh, & \tilde{c} &= \frac{\omega}{k}, & \tilde{c}_g &= \frac{\Omega}{K} = \frac{d\omega}{dk}, \\ c_0 &= \sqrt{gh}, & \tilde{c} &= \frac{\sigma}{\kappa} = c_0 \left(1 + \frac{\hat{a}}{2h} \right), & \kappa &= \frac{1}{h} \sqrt{\frac{3\hat{a}}{4h}}. \end{aligned}$$

k , K and κ are the wave numbers of the carrier wave, the envelope wave and the solitary wave respectively; ω , Ω and σ are their angular frequencies. A Stokes wave of constant amplitude is obtained by setting $e = K = \Omega = 0$.

The ‘symmetry’ of the solution (i.e. the same mathematical form for the carrier wave, for the envelope wave and for the solitary wave phases) is a criterion to consider that the re-introduction of the perturbation terms of (31) in the phases, is the right interpretation.

The solution holds three important non-dimensional parameters: $\tilde{\Gamma}$, $\bar{\Gamma}$ and $\hat{\Gamma}$ (Figure 2). These parameters are characteristics of the interaction and are only functions of the wave number k .

We have given here an approximate solution of the interaction between a solitary wave and a Stokes wave with sinusoidal modulations. The case in which the long wave is a cnoidal wave can be obtained, without any difficulty, by taking the corresponding solution of (10). The modulated Stokes wave is identical to the permanent solution of the linear Schrödinger equation. If we want to consider the more general case of the Stokes wave with non-linear modulations (like envelope solitons), it is necessary to introduce the short wave at order ε in the solution method. This operation disastrously increases the algebraic complexity of the resolution and, for this reason, it is not presented in this article (see Clamond, 1994). We have found that, in this case, the solution of the Stokes wave is the permanent solution of the cubic-Schrödinger equation. Excepted this

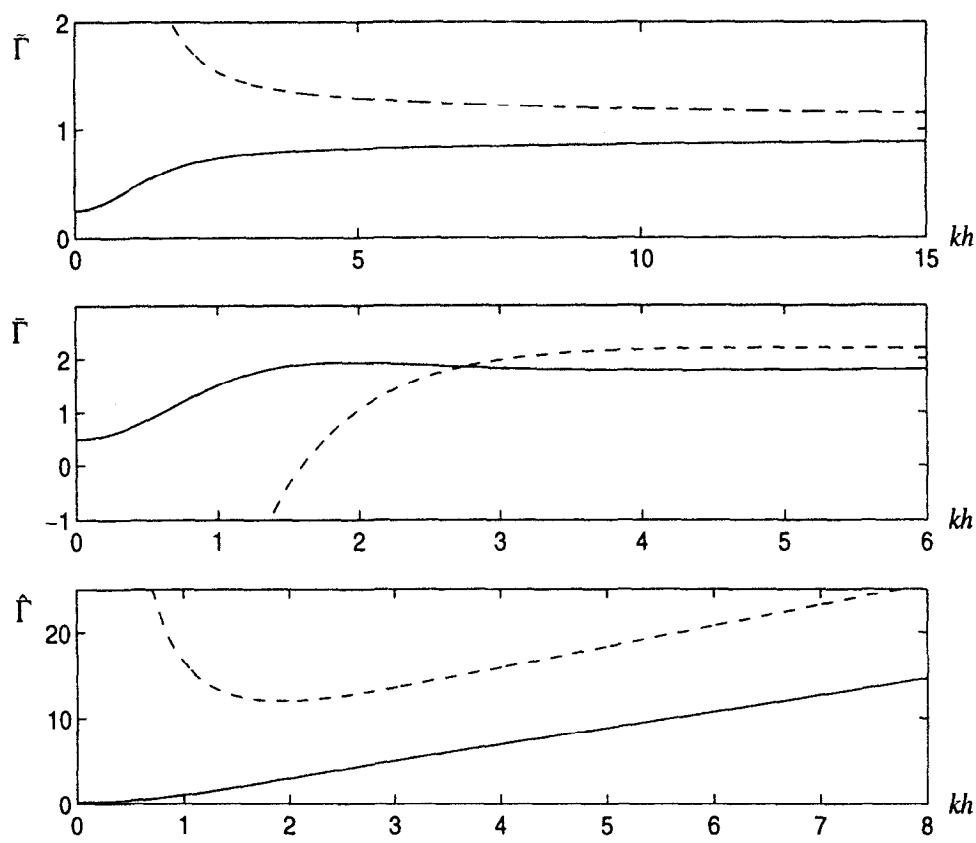


Fig. 2. – Characteristic parameters.
(—) weak interaction, (– –) strong interaction.

modification, the results are identical to the solution presented above. In particular, the phase forms and the characteristic parameters Γ are the same.

7. Phase shifts

7.1. PHASE SHIFT OF THE CARRIER WAVE

‘Far’ from the soliton, $\tilde{\phi}$ is constant. The short wave thus behaves like a Stokes wave. Although $\tilde{\phi}$ is constant at infinity, its value is not the same before and after the passage of the solitary wave: the phase of the short wave is changed. The phase shift is given by

$$\Delta\tilde{\phi} = \tilde{\phi}(+\infty) - \tilde{\phi}(-\infty) = \frac{4}{\sqrt{3}}kh\tilde{\Gamma}\sqrt{\frac{\tilde{a}}{h}}. \tag{32}$$

$\Delta\tilde{\phi}$ depends only on the amplitude of the solitary wave and on the wave number of the carrier wave, but not on the amplitude of the latter, just like in the simplified formula. There too, the phase shift is proportional to the square root of the amplitude of the soliton, but the proportion factor is not the same as for the simplified relation.

In solving the equations we assumed that the solitary wave came from infinity upstream. For the case where the short wave comes from infinity downstream (the two waves propagating in opposite directions), one merely

has to set ω positive. If we wish to investigate the case where the two waves propagate in the same direction, ω should be taken to be negative.

In figure 3 the variation of the phase shift is shown as a function of the wave number. It can be seen that $\Delta\tilde{\phi}$ is larger when the waves propagate in the same direction than when they propagate in opposite directions. This result differs from the simplified formula, which predicts identical phase shifts.

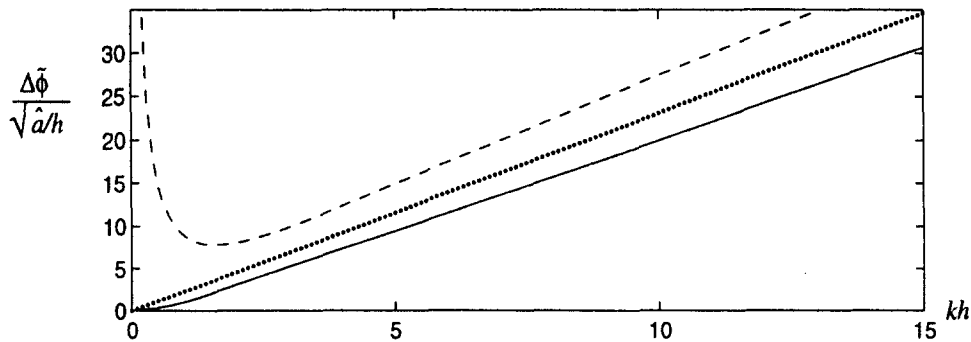


Fig. 3. – Phase shift of the carrier wave.
(...) simplified formula, (—) weak interaction, (– –) strong interaction.

The simplified formula, for a given soliton amplitude, always under-estimates the phase shift when the waves propagate in the same direction, and overestimates it in the opposite case. If ω is positive, $\Delta\tilde{\phi}$ is an increasing function of k . If ω is negative, $\Delta\tilde{\phi}$ decreases at first, then increases with k . These results show that the relative direction of propagation of the two waves has a quantitative and qualitative influence on the results. In what follows, we shall denote as strong interaction the case where the waves propagate in the same direction and weak interaction for the opposite case.

In the limiting case where the Stokes wave has a very small wavelength (k and ω are large), the phase shift tends to the asymptotic formula (for ω either positive or negative)

$$\Delta\tilde{\phi} \sim \frac{4}{\sqrt{3}} kh \sqrt{\frac{\hat{a}}{h}}. \quad (33)$$

This gives us again the simplified formula (5). This result shows that the shorter the Stokes wave becomes, the more the interaction can be considered to be linear in k . (33) is an asymptotic direction for (32) but not an asymptotic limit. To see this, we note that

$$\lim_{k \rightarrow +\infty} \Delta\tilde{\phi} - \frac{4}{\sqrt{3}} kh \sqrt{\frac{\hat{a}}{h}} = \begin{cases} -\infty & \text{if } \omega > 0 \\ +\infty & \text{if } \omega < 0 \end{cases}.$$

This means that the phase shift given by the theoretical formula (32), does not get closer to the simplified formula (5) at large wave numbers; on the contrary, they move increasingly apart, while taking the same direction (Figure 3).

We now consider the opposite limit where the Stokes wave is long (small k). If the waves travel in opposite directions, the theoretical and simplified formulae are equivalent. This is not true for waves traveling in the same direction, where the simplified formula predicts a very small phase shift, while the theoretical formula predicts an infinite phase shift. Nonetheless, these results certainly have no physical reality. To establish the theoretical

formula we assumed that the phase of the short wave varies ‘fast’. But if k is small, the phase variations are ‘slow’ and the assumptions for our calculations are therefore invalid. We therefore place no reliance on the results predicted by (32) in the case of small k .

Given these observations, we can conclude that the simplified formula for the phase shift has no domain of validity. This fact was confirmed by experimental investigations (see section 9), where a good agreement between the theoretical formula and the experiments was found.

7.2. PHASE SHIFT OF THE ENVELOPE WAVE

As well as the carrier wave, the envelope wave is shifted by the soliton. The phase shift is

$$\Delta\bar{\phi} = \bar{\phi}(+\infty) - \bar{\phi}(-\infty) = \frac{4}{\sqrt{3}}Kh\bar{\Gamma}\sqrt{\frac{\hat{a}}{h}}. \quad (34)$$

The shift of the envelope wave is different than the shift of the carrier wave. In the weak interaction the shift is always positive, hence the envelope wave is shifted in the same direction as the carrier wave (Figure 2). On the other hand, in the strong interaction for low wave numbers, the shift is negative, hence the envelope wave is shifted in the opposite direction of the carrier wave! To understand this phenomenon, we rewrite η in the form

$$\begin{aligned} \eta &= \hat{a}\sqrt{1+e\sin 2\bar{S}}\sin(\hat{S}+\psi) + \hat{a}\operatorname{sech}^2\hat{S}_0 \\ &= \hat{a}_1\sin(\hat{S}_1) + \hat{a}_2\sin(\hat{S}_2) + \hat{a}\operatorname{sech}^2\hat{S}_0, \end{aligned}$$

with

$$\begin{aligned} \hat{S}_1 &= k_1x + \omega_1t + \tilde{\delta}_1 + \phi_1, & \hat{S}_2 &= k_2x + \omega_2t + \tilde{\delta}_2 + \phi_2, \\ \phi_1 &= \tilde{\phi} + \bar{\phi}, & \phi_2 &= \tilde{\phi} - \bar{\phi}, & \tilde{a}_1 &= \hat{a}\sqrt{\frac{1+\sqrt{1-e^2}}{2}}, & \tilde{a}_2 &= \hat{a}\sqrt{\frac{1-\sqrt{1-e^2}}{2}}, \\ k_1 &= k + K, & \omega_1 &= \omega + \Omega, & \tilde{\delta}_1 &= \tilde{\delta} + \bar{\delta} + \frac{1}{2}\arcsin e, \\ k_2 &= k - K, & \omega_2 &= \omega - \Omega, & \tilde{\delta}_2 &= \tilde{\delta} - \bar{\delta} + \frac{1}{2}\arcsin e. \end{aligned}$$

Hence, this modulated Stokes wave is the superposition of two Stokes waves of constant amplitudes. With our notations $k_1 > k_2$, $\omega_1 > \omega_2$ and $\bar{\phi} = (\phi_1 - \phi_2)/2$. In the weak interaction, the phase shifts increase with k (Figure 2), and hence the shorter component of the modulated wave suffers a more important shift than its longer component. The difference $\phi_1 - \phi_2$ is then always positive: the envelope wave is shifted in the same direction as the carrier wave. On the other hand, in the strong interaction for low wave numbers, the phase shifts decrease with k and the difference $\phi_1 - \phi_2$ can be negative: the envelope is shifted in the opposite direction of the carrier wave one. Hence, although every component of a modulated Stokes wave is shifted in the same direction, the envelope can be shifted in the opposite direction.

7.3. PHASE SHIFT OF THE SOLITARY WAVE

The short wave is present everywhere in space and time. Thus the soliton does not suffer a definitive phase shift but a perpetual modification. In a characteristic length of the soliton, there are many periods of the Stokes wave. Depending on the direction of the velocity induced by the short wave, the soliton phase moves forwards or backwards: the soliton folds (Figure 4, the effect is exaggerated on the figure to make it more visible).

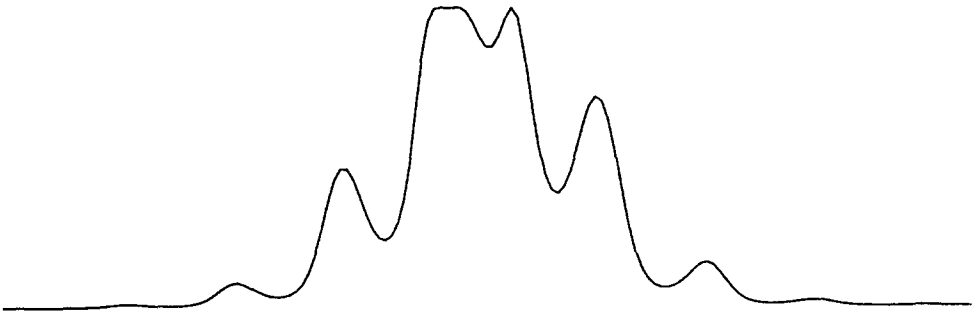


Fig. 4. – Modified soliton by Stokes wave action.

The soliton’s phase is modified by the velocity potential of the short wave which has significant vertical variations, therefore iso-phases are not vertical lines. For an iso-phase, \hat{S} is constant hence

$$\frac{d\hat{S}}{dy} = \frac{\partial \hat{S}}{\partial y} + \frac{\partial \hat{S}}{\partial x} \frac{dx}{dy} = 0 \quad \text{if } \hat{S} = \text{constant},$$

and with the definition of \hat{S} we deduced that on the bottom

$$\frac{dx}{dy} = 0 \quad \text{at} \quad y = -h.$$

All the iso-phases are perpendicular to the bottom (Figure 5).

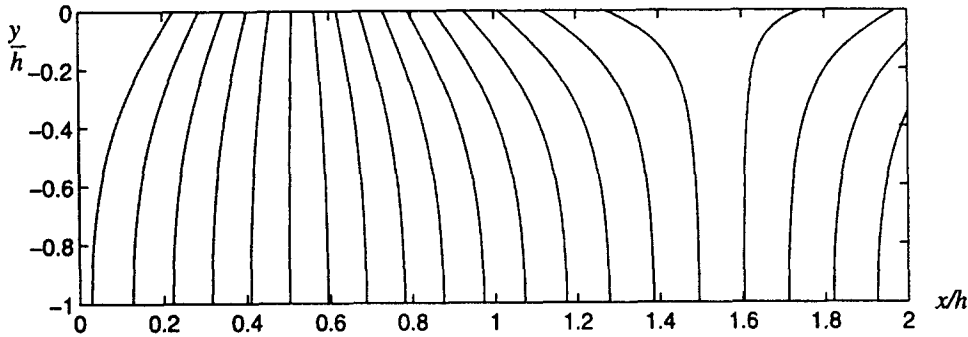


Fig. 5. – Vertical variations of soliton’s iso-phases.

The phase modifications of the long wave are more important near the surface and are characterized by the parameter $\hat{\Gamma}$. We can see on figure 2 that the magnitude of $\hat{\Gamma}$ is greater for the strong interaction than for the weak interaction. The phase modifications of the soliton are proportional to the Stokes wave amplitude which is necessarily very small for stability reasons. However $\hat{\Gamma}$ can take important values and the action of the short wave can be significant.

8. Doppler effects

8.1. DOPPLER EFFECT ON THE STOKES WAVE

In investigating the phase shifts, we examined permanent modifications to the short wave on either side of the solitary wave. We now study the modifications to the short wave ‘on’ the soliton. To do this we introduce the notions of local wave numbers and angular frequencies defined by

$$\tilde{k} \equiv \frac{\partial \tilde{S}}{\partial x} = k + \frac{k\hat{a}\tilde{\Gamma}}{h} \operatorname{sech}^2(\kappa x - \sigma t + \hat{\delta}), \quad (35)$$

$$\tilde{\omega} \equiv \frac{\partial \tilde{S}}{\partial t} = \omega - \hat{c} \frac{k\hat{a}\tilde{\Gamma}}{h} \operatorname{sech}^2(\kappa x - \sigma t + \hat{\delta}), \quad (36)$$

$$\tilde{K} \equiv \frac{\partial S}{\partial x} = K + \frac{K\hat{a}\tilde{\Gamma}}{h} \operatorname{sech}^2(\kappa x - \sigma t + \hat{\delta}), \quad (37)$$

$$\tilde{\Omega} \equiv \frac{\partial S}{\partial t} = \Omega - \hat{c} \frac{K\hat{a}\tilde{\Gamma}}{h} \operatorname{sech}^2(\kappa x - \sigma t + \hat{\delta}), \quad (38)$$

‘Far’ from the soliton, \tilde{k} , \tilde{K} , $\tilde{\omega}$ and $\tilde{\Omega}$ coincide with k , K , ω and Ω . As the short wave travels over the soliton, it is subjected to a continuous change in its wave number and its frequency. These are the Doppler effects. This change reaches a maximum at the top of the waves. Eliminating the function sech^2 from relations (35–38) we get

$$\tilde{\omega} + \hat{c}\tilde{k} = \omega + \hat{c}k, \quad \tilde{\Omega} + \hat{c}\tilde{K} = \Omega + \hat{c}K. \quad (39)$$

These relations states that the quantities $\tilde{\omega} + \hat{c}\tilde{k}$ and $\tilde{\Omega} + \hat{c}\tilde{K}$ are conserved throughout the interaction. If we know the variation of the wave numbers, we then know that of the frequencies (and *vice versa*). Thus we only need to investigate the wave numbers.

$\tilde{\omega} + \hat{c}\tilde{k}$ and $\tilde{\Omega} + \hat{c}\tilde{K}$ are the angular frequencies of the carrier wave and the envelope wave observed in the reference frame of the soliton. (39) simply states that in this frame the short wave frequencies are constant. The wave numbers, however, vary only in space (in the frame of the soliton). They are larger ‘on’ the soliton than ‘far’ from it. The soliton contracts the short wave. The effect of the soliton is to increase the local Stokes wave numbers. The increase is greater if the Stokes wave travels in the same direction as the soliton than if they travel in opposite directions. These effects can be characterized by considering the maximum relative variations of the wave numbers, normalized by the amplitude of the soliton. These quantities are given by the relations

$$\frac{\Delta \tilde{k}}{k\hat{a}/h} \equiv \frac{\tilde{k}(0) - \tilde{k}(\infty)}{\tilde{k}(\infty)\hat{a}/h} = \tilde{\Gamma}, \quad \frac{\Delta \tilde{K}}{K\hat{a}/h} \equiv \frac{\tilde{K}(0) - \tilde{K}(\infty)}{\tilde{K}(\infty)\hat{a}/h} = \tilde{\Gamma}. \quad (40)$$

Hence, the parameters $\tilde{\Gamma}$ and $\tilde{\Gamma}$ are related to the maximum Doppler effects.

The increase in wave number causes an increase in another parameter: the steepness $S = \hat{a}\tilde{k}$ (for a short wave with a constant amplitude). The stability of the wave is conditioned by the value of this parameter. Below a critical value of steepness the wave is stable, while above this value the wave breaks. Hence if the Doppler effect is too large the steepness can increase beyond the critical threshold and cause the wave to break. In certain experimental trials we have indeed observed the short wave breaking over the soliton. Since the amplitude of the short wave is constant the relative variations of steepness are equal to those of the wave number (formula (40) and figure 2). The Doppler effect also reveals the qualitative and quantitative differences between the strong and the weak interactions.

8.2. DOPPLER EFFECT ON THE LONG WAVE

For simplicity, we will only consider the action on the long wave due to a short wave of constant amplitude. As for the short wave, we can consider the local wave number and frequency of the long wave

$$\hat{\kappa} \equiv \frac{\partial \hat{S}}{\partial x} = \kappa + \frac{kg\tilde{a}\kappa\hat{\Gamma}}{c_0\omega} \frac{\cosh k(y+h)}{\cosh kh} \sin(kx + \omega t + \tilde{\delta}), \quad (41)$$

$$\hat{\sigma} \equiv -\frac{\partial \hat{S}}{\partial t} = \sigma - \frac{g\tilde{a}\kappa\hat{\Gamma}}{c_0} \frac{\cosh k(y+h)}{\cosh kh} \sin(kx + \omega t + \tilde{\delta}). \quad (42)$$

Eliminating the sin-function from relations (41–42) we get

$$\hat{\sigma} + \tilde{c}\hat{\kappa} = \sigma + \tilde{c}\kappa. \quad (43)$$

This relation states that the quantity $\hat{\sigma} + \tilde{c}\hat{\kappa}$ is conserved throughout the interaction. $(\hat{\sigma} + \tilde{c}\hat{\kappa})$ is the angular frequency of the soliton observed in the reference frame of Stokes wave. The wave number, however, varies only in space. The soliton undergoes a periodic Doppler effect which is more important at the surface. The relative maximum variations of the soliton's wave number can be defined by

$$\frac{\Delta\hat{\kappa}}{\kappa\tilde{a}/h} \equiv \frac{\max(\hat{\kappa}) - \kappa}{\kappa\tilde{a}/h} = \frac{c_0 k \hat{\Gamma}}{\omega} \frac{\cosh k(y+h)}{\cosh kh}. \quad (44)$$

This quantity at the free surface and at the bottom, is plotted on figure 6.

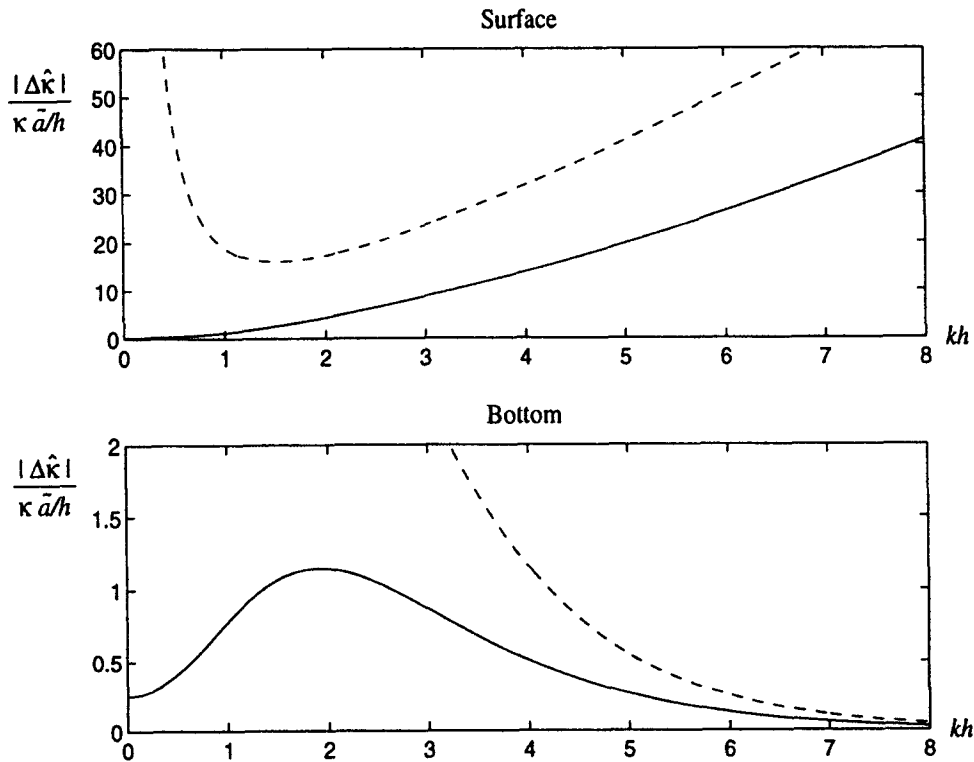


Fig. 6. – Doppler effect of the soliton
(—) weak interaction, (– –) strong interaction.

Like the other effects, the Doppler effect on the soliton is greater for the strong interaction. Of course the Doppler effect is more important at the surface. For a very short wave (i.e. of infinite depth) there is logically no Doppler effect on the soliton at the bottom because the velocity induced by the Stokes wave is very small.

9. Experimental investigation

The experiments were performed in the 36 m glass wave tank at the L.E.G.I. (Figure 7). The width of the tank is 0.55 m. At one end it has a piston wave-maker that can generate solitary waves, while at the other end a wedge wave-maker generates short waves. The experiment consists in generating a wave train of short wavelength and a soliton moving in the opposite direction, with various amplitudes of the soliton and frequencies of the short wave. The amplitude of the latter is kept constant at about 5 mm. The experimental conditions selected for our experiments are: depth of water at rest 255 mm, two frequencies of short waves: 1.5 Hz and 2.3 Hz. The relative amplitudes of the solitons was varied from 0.05 to 0.7. For each experiment, we have compared the phase of the short wave before and after the soliton (Figure 8). More details of these experiments can be found in Clamond (1994) and in Clamond and Barthélemy (1995).

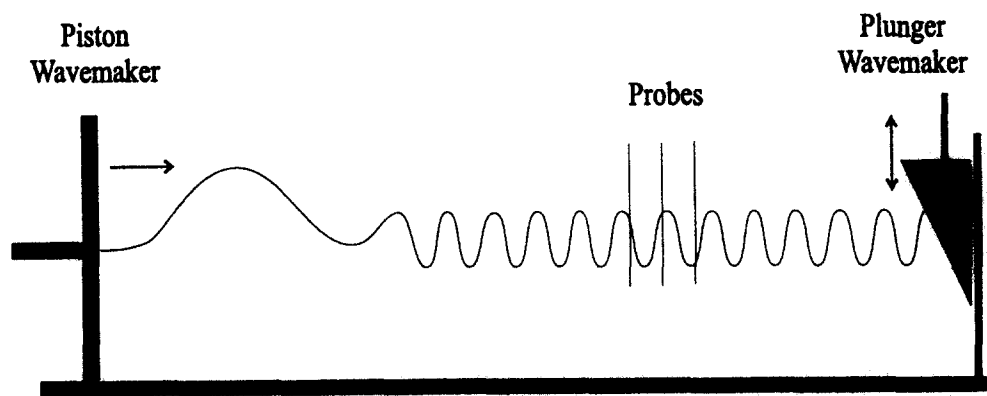


Fig. 7. – Experimental tank.

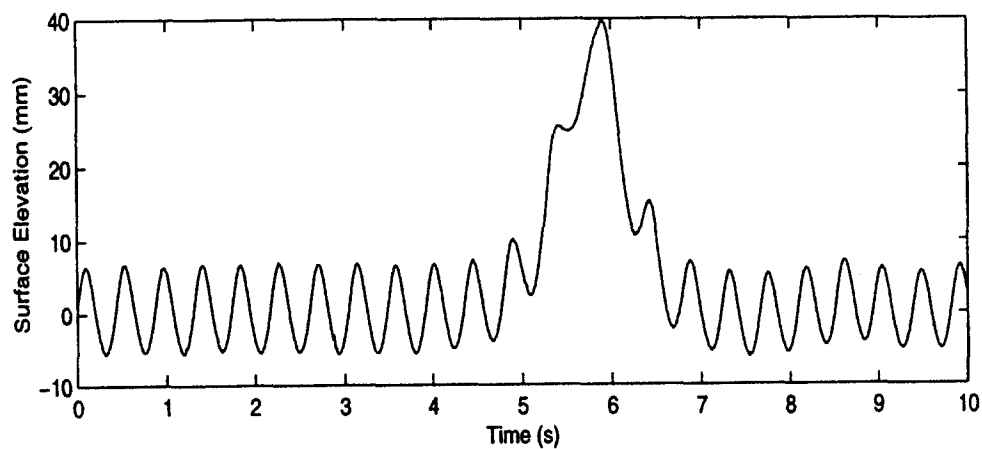


Fig. 8. – An interaction example.

The piston wave-maker has imperfections, and in addition generates a dispersive wave train following the solitary wave, the amplitude of which is roughly 10% of that of the soliton. The wedge wave-maker undergoes pure sinusoidal motion. Harmonic analysis of waves generated by this wave-maker indicates that the amplitude of the first harmonic is ten times smaller than that of the fundamental. To make phase shift measurements with precision, we used a method based on harmonic analysis. The amplitude of the short wave and their phases before and after the soliton are determined by a least square minimization of a trigonometric polynomial fit (Clamond and Barthélemy, 1995). Tests show that white noise affects the result very little. In contrast, if the wave is amplitude modulated, the signal is no longer harmonic, and errors can be large. Measurements were carried out using five probes. Each trial thus yields five measurements of the phase shift. The spread of values from the different probes ranges from a few degrees for small amplitude solitons to 40 degrees for large solitons. This result shows that the technique of phase measurement is intrinsically precise.

The phase shifts predicted by the theoretical formula and the simplified formula has the form

$$\Delta\tilde{\phi} = \zeta \sqrt{\hat{a}/h}, \quad (45)$$

where the proportional constant ζ is completely determined by the frequency of the short wave. It is different for the two formulae. Over a series of experiments, the short wave generating apparatus drifts. For a given setting, therefore, two consecutive trials do not generate a short wave of exactly the same frequency; the fluctuations are about 0.05 Hz, which is fairly small. However, as already stated, the phase is a rapidly varying quantity, and a small deviation in frequency can lead to a large variation in phase shift. In formula (45), this shows up as a large variation in the constant ζ . $\Delta\tilde{\phi}$ thus varies as a function of two parameters (\hat{a} and ω). It is more rewarding therefore to investigate the *relative* phase shift $\chi = \Delta\tilde{\phi}/\zeta$, which depends only on the single parameter \hat{a} .

The experimental results are displayed in figure 9. This shows two graphs corresponding to the two frequencies chosen for the short wave. It can be seen from these results that, within an acceptable error, the phase shift is indeed proportional to the square root of the soliton amplitude. For the proportional constant, the experimental evidences are favorable of the theoretical formula in both trials. The investigation yielded results only for the weak interaction situation. A strong interaction (the solitary wave and high frequency wave train propagating in the same direction) requires the waves to be generated at the same end of the tank, which introduces practical difficulties.

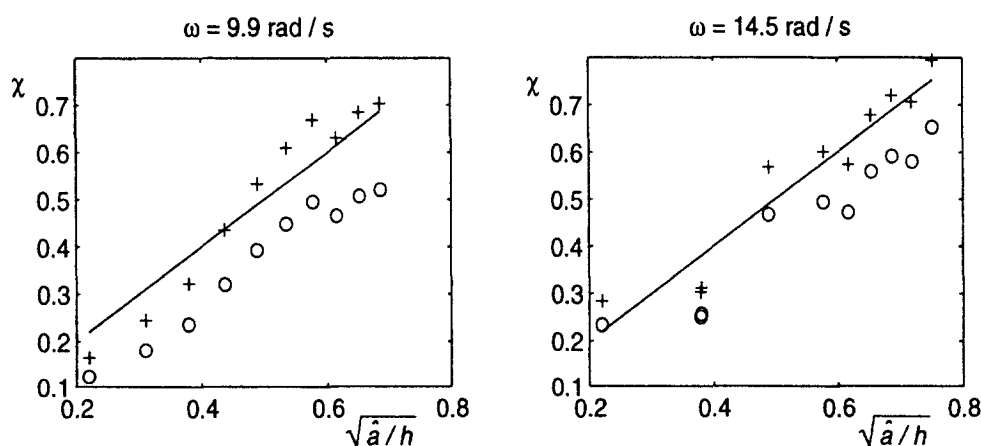


Fig. 9. – Phase shift measurements.
(—) $\chi = \sqrt{\hat{a}/h}$, (+) theoretical formula, (o) simplified formula.

10. Conclusion

In this investigation we show the changes experienced by a Stokes wave interacting with a solitary wave. After some heavy calculations we obtain a simple explicit analytical solution. The physical interpretation given to certain terms arising in the solution of the equations is qualitatively different from that of Chu and Mei (1970) and Shen *et al.* (1994). The phase shifts, the Doppler effects and the vertical variations of the long wave phase display qualitative and quantitative differences between the weak and the strong interactions. The wave numbers and the local angular frequencies are connected by conservation relations that are independent of that connecting the overall quantities. The experimental part of this work provides measurements of the phase shift experienced by the short wave. These measurements corroborate the theoretical formula, to the detriment of the simplified formula. The results generally become more marked as the Stokes waves become shorter. The opposite situation is inconsistent with our working hypotheses, and the predictions could be unrealistic. To treat the case of interaction between two long waves, the shallow water theory must be used.

More complete experimental investigations are necessary to verify the theoretical predictions in the case of the strong interaction. Some numerical investigations should also be used. We hope that this work, especially the results concerning the modifications of the long wave, will be useful in wave interaction problems to describe the phenomena with a different point of view.

A. Appendix

The resolution of the Laplace equation yields the expression of every $F_{n,p}$

$$\begin{aligned}
 F_{3,0} &= -\left(\frac{y^2}{2} + y\right) \frac{\partial^2 A_{1,0}}{\partial \alpha^2} + A_{3,0}, \\
 F_{3,1} &= A_{3,1} \cosh k(y+h) - i(y+h) \sinh k(y+h) \frac{\partial A_{2,1}}{\partial \alpha}, \\
 F_{4,0} &= -\left(\frac{y^2}{2} + y\right) \frac{\partial^2 A_{2,0}}{\partial \alpha^2} + A_{4,0}, \\
 F_{4,1} &= A_{4,1} \cosh k(y+h) - \frac{1}{2}(y+h)^2 \cosh k(y+h) \frac{\partial^2 A_{2,1}}{\partial \alpha^2} \\
 &\quad - i(y+h) \sinh k(y+h) \left(\frac{\partial A_{3,1}}{\partial \alpha} + iA_{2,1} \frac{\partial S_2}{\partial \alpha} \right), \\
 F_{4,2} &= A_{4,2} \cosh 2k(y+h), \\
 F_{5,0} &= \left(\frac{y^4}{24} + \frac{y^3}{6} - \frac{y}{3}\right) \frac{\partial^4 A_{1,0}}{\partial \alpha^4} - \left(\frac{y^2}{2} + y\right) \frac{\partial^2 A_{3,0}}{\partial \alpha^2} + A_{5,0}, \\
 F_{5,1} &= i \sinh k(y+h) \left[\frac{(y+h)^3}{6} \frac{\partial^3 A_{2,1}}{\partial \alpha^3} - (y+h) \left(\frac{\partial A_{4,1}}{\partial \alpha} + iA_{3,1} \frac{\partial S_2}{\partial \alpha} + iA_{2,1} \frac{\partial S_3}{\partial \alpha} \right) \right] \\
 &\quad + \cosh k(y+h) \left[A_{5,1} - \frac{(y+h)^2}{2} \left(\frac{\partial^2 A_{3,1}}{\partial \alpha^2} + iA_{2,1} \frac{\partial^2 S_2}{\partial \alpha^2} + 2i \frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial S_2}{\partial \alpha} \right) \right],
 \end{aligned}$$

where $A_{n,p} \equiv A_{n,p}(\alpha, \tau)$ are undetermined functions. Their determination needs to solve the equations at the free surface. It is easy to demonstrate, by a change of definition in S of order ε^2 , that the argument of $A_{3,1}$ can be taken to be equal to that of $A_{2,1}$ with no loss of generality to the solution. Indeed, if we consider the

approximation to order ε^3 for the velocity potential, we have

$$\varphi = \varepsilon F_{1,0} + \varepsilon^2 F_{2,0} + \varepsilon^3 F_{3,0} + [\varepsilon^2 F_{2,1} + \varepsilon^3 F_{3,1}] \exp\left(i \frac{S}{\varepsilon}\right) + O(\varepsilon^4) + \text{c.c.}$$

At this point S_2 is not yet determined by the equations, we are therefore free to set

$$S_2 = S'_2 + S''_2, \quad S' = S - \varepsilon^2 S''_2, \quad A_{3,1} = A'_{3,1} - i A_{2,1} S''_2.$$

Using the definitions of the $F_{2,1}$ and $F_{3,1}$, and expanding $\exp(i\varepsilon S''_2)$ as a Taylor series, we obtain

$$\begin{aligned} \varphi = \varepsilon^2 & \left[(A_{2,1} + \varepsilon A'_{3,1}) \cosh k(y+h) - i\varepsilon(y+h) \sinh k(y+h) \frac{\partial A_{2,1}}{\partial \alpha} \right] \exp\left(i \frac{S'}{\varepsilon}\right) \\ & + \varepsilon F_{1,0} + \varepsilon^2 F_{2,0} + \varepsilon^3 F_{3,0} + O(\varepsilon^4) + \text{c.c.} \end{aligned}$$

S''_2 can be chosen completely arbitrarily with no loss of generality to the solution. We therefore choose

$$S''_2 = -\frac{|A_{3,1}|}{|A_{2,1}|} \sin[\arg(A_{3,1}) - \arg(A_{2,1})].$$

The argument of $A'_{3,1}$ is then equal to the argument of $A_{2,1}$. We have thus shown, through a change of order ε^2 in the definition of S , that the argument of $A_{3,1}$ can be taken equal to that of $A_{2,1}$ with no loss of generality for the solution.

By the same reasoning, it can be shown through a change of order ε^3 in the definition of S , that the argument of $A_{4,1}$ is equal to that of $A_{2,1}$. By recurrence it is shown that all the $A_{n,1}$ have the same argument ψ . This fact is essential to achieve the resolution: without this consideration the problem is under-determined.

The resolution of the Bernoulli equation gives the expressions of the surface elevation coefficients $G_{n,p}$ as functions of the velocity potential coefficients $A_{n,p}$

$$\begin{aligned} G_{3,0} &= -\frac{1}{g} \frac{\partial A_{2,0}}{\partial \tau}, \\ G_{3,1} &= -\frac{i\omega}{g} \cosh kh A_{3,1} - \frac{h\omega}{g} \sinh kh \frac{\partial A_{2,1}}{\partial \alpha} - \frac{1}{g} \cosh kh \frac{\partial A_{2,1}}{\partial \tau}, \\ G_{4,0} &= -\frac{1}{g} \frac{\partial A_{3,0}}{\partial \tau} - \frac{1}{2g} \left(\frac{\partial A_{1,0}}{\partial \alpha} \right)^2 - \frac{k^2}{g} A_{2,-1} A_{2,1}, \\ G_{4,1} &= -\frac{h\omega}{g} \sinh kh \left(\frac{\partial A_{3,1}}{\partial \alpha} + i A_{2,1} \frac{\partial S_2}{\partial \alpha} \right) - \frac{1}{g} \cosh kh \left(\frac{\partial A_{3,1}}{\partial \tau} + i A_{2,1} \frac{\partial S_2}{\partial \tau} \right) - \frac{i\omega}{g} \cosh kh A_{4,1} \\ &\quad + \frac{ih^2\omega}{2g} \cosh kh \frac{\partial^2 A_{2,1}}{\partial \alpha^2} + \frac{ih}{g} \sinh kh \frac{\partial^2 A_{2,1}}{\partial \alpha \partial \tau} + \frac{ik}{g} \left(\frac{\omega}{g} \sinh kh \frac{\partial A_{1,0}}{\partial \tau} - \cosh kh \frac{\partial A_{1,0}}{\partial \alpha} \right) A_{2,1}, \\ G_{4,2} &= -\frac{2i\omega}{g} \cosh 2kh A_{4,2} + \frac{k^2}{g} \left(\frac{1}{2} - \sinh^2 kh \right) A_{2,1}^2, \end{aligned}$$

$$\begin{aligned}
G_{5,1} = & -\frac{h\omega}{g} \sinh kh \left(\frac{\partial A_{4,1}}{\partial \alpha} + iA_{3,1} \frac{\partial S_2}{\partial \alpha} + iA_{2,1} \frac{\partial S_3}{\partial \alpha} - \frac{h^2}{6} \frac{\partial^3 A_{2,1}}{\partial \alpha^3} - \frac{1}{gh} \frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial A_{1,0}}{\partial \tau} \right) \\
& - \frac{1}{g} \cosh kh \left(\frac{\partial A_{4,1}}{\partial \tau} + iA_{3,1} \frac{\partial S_2}{\partial \tau} + iA_{2,1} \frac{\partial S_3}{\partial \tau} - \frac{h^2}{2} \frac{\partial^3 A_{2,1}}{\partial \alpha^2 \partial \tau} + \frac{1}{\sqrt{gh}} \frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial A_{1,0}}{\partial \alpha} \right) \\
& + \frac{ih^2\omega}{2g} \cosh kh \left(\frac{\partial^2 A_{3,1}}{\partial \alpha^2} + iA_{2,1} \frac{\partial^2 S_2}{\partial \alpha^2} + 2i \frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial S_2}{\partial \alpha} \right) - \frac{i\omega}{g} \cosh kh A_{5,1} \\
& + \frac{ih}{g} \sinh kh \left(\frac{\partial^2 A_{3,1}}{\partial \alpha \partial \tau} + iA_{2,1} \frac{\partial^2 S_2}{\partial \alpha \partial \tau} + i \frac{\partial A_{2,1}}{\partial \tau} \frac{\partial S_2}{\partial \alpha} + i \frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial S_2}{\partial \tau} \right) \\
& + \frac{i\omega k}{g^2} \sinh kh \left(A_{2,1} \frac{\partial A_{2,0}}{\partial \tau} + A_{3,1} \frac{\partial A_{1,0}}{\partial \tau} \right) - \frac{ik}{g} \cosh kh \left(A_{2,1} \frac{\partial A_{2,0}}{\partial \alpha} + A_{3,1} \frac{\partial A_{1,0}}{\partial \alpha} \right) \\
& - \frac{kh}{g} \sinh kh \left(\frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial A_{1,0}}{\partial \alpha} - \frac{1}{gh} \frac{\partial A_{2,1}}{\partial \tau} \frac{\partial A_{1,0}}{\partial \tau} - A_{2,1} \frac{\partial^2 A_{1,0}}{\partial \alpha^2} \right) + \frac{\omega kh}{g^2} \cosh kh \frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial A_{1,0}}{\partial \tau}.
\end{aligned}$$

The impermeability of the free surface gives in turn

$$\begin{aligned}
i \left(\frac{\partial A_{3,1}}{\partial \tau} - \tilde{c}_g \frac{\partial A_{3,1}}{\partial \alpha} \right) &= \left(\frac{\partial S_2}{\partial \tau} - \tilde{c}_g \frac{\partial S_2}{\partial \alpha} \right) A_{2,1} - h \tanh kh \frac{\partial^2 A_{2,1}}{\partial \alpha \partial \tau} \\
&+ \frac{1}{2\omega} \left(gh \frac{\partial^2 A_{2,1}}{\partial \alpha^2} - \frac{\partial^2 A_{2,1}}{\partial \tau^2} \right) + k \left(\frac{\partial A_{1,0}}{\partial \alpha} + \frac{\omega}{g \sinh 2kh} \frac{\partial A_{1,0}}{\partial \tau} \right) A_{2,1},
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\frac{\partial^2 A_{3,0}}{\partial \tau^2} - gh \frac{\partial^2 A_{3,0}}{\partial \alpha^2} &= \frac{gh^3}{3} \frac{\partial^4 A_{1,0}}{\partial \alpha^4} - \frac{\partial A_{1,0}}{\partial \tau} \frac{\partial^2 A_{1,0}}{\partial \alpha^2} - 2 \frac{\partial A_{1,0}}{\partial \alpha} \frac{\partial^2 A_{1,0}}{\partial \alpha \partial \tau} \\
&- k^2 \frac{\partial |A_{2,1}|^2}{\partial \tau} - 2\omega k \cosh^2 kh \frac{\partial |A_{2,1}|^2}{\partial \alpha}.
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
i \left(\frac{\partial A_{4,1}}{\partial \tau} - \tilde{c}_g \frac{\partial A_{4,1}}{\partial \alpha} \right) &= \left(\frac{\partial S_2}{\partial \tau} - \tilde{c}_g \frac{\partial S_2}{\partial \alpha} \right) A_{3,1} + \left(\frac{\partial S_3}{\partial \tau} - \tilde{c}_g \frac{\partial S_3}{\partial \alpha} \right) A_{2,1} - \frac{ik}{\omega} \frac{\partial A_{2,1}}{\partial \tau} \frac{\partial A_{1,0}}{\partial \alpha} \\
&+ \frac{1}{2\omega} \left(gh \frac{\partial^2 A_{3,1}}{\partial \alpha^2} - \frac{\partial^2 A_{3,1}}{\partial \tau^2} + iA_{2,1} \left(gh \frac{\partial^2 S_2}{\partial \alpha^2} - \frac{\partial^2 S_2}{\partial \tau^2} \right) + 2igh \frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial S_2}{\partial \alpha} - 2i \frac{\partial A_{2,1}}{\partial \tau} \frac{\partial S_2}{\partial \tau} \right) \\
&+ \frac{i\omega^2}{g^2} \frac{\partial A_{2,1}}{\partial \tau} \frac{\partial A_{1,0}}{\partial \tau} - h \tanh kh \left(\frac{\partial^2 A_{3,1}}{\partial \alpha \partial \tau} + iA_{2,1} \frac{\partial^2 S_2}{\partial \alpha \partial \tau} + i \frac{\partial A_{2,1}}{\partial \tau} \frac{\partial S_2}{\partial \alpha} + i \frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial S_2}{\partial \tau} \right) \\
&+ \frac{ih}{2} \left(h \frac{\partial^3 A_{2,1}}{\partial \alpha^2 \partial \tau} + \frac{\tilde{c}}{g} \frac{\partial^3 A_{2,1}}{\partial \tau^2 \partial \alpha} - \frac{h}{3} (\tilde{c}_g + \tilde{c}) \frac{\partial^3 A_{2,1}}{\partial \alpha^3} \right) + i \left(\frac{\omega^3}{2kg^2} - \frac{k}{\omega} \right) \frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial A_{1,0}}{\partial \tau} \\
&+ k \left(\frac{\partial A_{1,0}}{\partial \alpha} + \frac{\omega}{g \sinh 2kh} \frac{\partial A_{1,0}}{\partial \tau} \right) A_{3,1} + k \left(\frac{\partial A_{2,0}}{\partial \alpha} + \frac{\omega}{g \sinh 2kh} \frac{\partial A_{2,0}}{\partial \tau} \right) A_{2,1} \\
&+ i \left(\frac{\omega^2}{2g^2} \frac{\partial^2 A_{1,0}}{\partial \tau^2} - \frac{k}{\omega} \frac{\partial^2 A_{1,0}}{\partial \alpha \partial \tau} + \frac{h\omega^2 - g}{2g} \frac{\partial^2 A_{1,0}}{\partial \alpha^2} \right) A_{2,1} - i \left(\frac{g + h\omega^2}{g} \right) \frac{\partial A_{2,1}}{\partial \alpha} \frac{\partial A_{1,0}}{\partial \alpha}.
\end{aligned} \tag{A.3}$$

We have introduced in the text an intermediate function P :

$$\begin{aligned}
P \equiv & \frac{i}{2} \frac{\omega_{kk}}{\tilde{c} + \tilde{c}_g} \frac{d^2 A_{3,1}}{d\mu^2} - \frac{i}{h} R_2 A_{3,1} + \frac{h^2}{6} \left(\frac{2\tilde{c}_g - \tilde{c} + 3\tilde{c}\tilde{c}_g^2/\tilde{c}^2}{\tilde{c} + \tilde{c}_g} \right) \frac{d^3 A_{2,1}}{d\mu^3} \\
&+ \frac{\tilde{c}}{h\omega} \left(1 + k^2 h^2 \frac{\tilde{c} - \tilde{c}_g}{\tilde{c} + \tilde{c}_g} \frac{\tilde{c}^3}{\tilde{c}^3} \right) \frac{d(A_{2,1} R_2)}{d\mu}.
\end{aligned}$$

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